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Maps of Ostaszewski and related spaces

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Abstract

Under $\diamondsuit$ there is an Ostaszewski space which is retractive, homeomorphic to every uncountable closed subspace, and homeomorphic to every locally countable regular Hausdorff uncountable continuous image, and also an Ostaszewski space with none of these properties. There is a ZFC example of a thin-tall locally compact scattered space for which each nonempty Cantor–Bendixson remainder is a retract. Attention is also paid to robustness under various types of forcing.

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0. Introduction

Are there any superatomic Boolean algebras which are

(a) retractive?

(b) isomorphic to every uncountable homomorphic image?

(c) isomorphic to every uncountable subalgebra?

Under $\diamondsuit$ the answer to (a) and (b) was shown to be "yes" by Shelah, and the answer to (c) was shown to be "yes" by Bonnet and Rubin. Both constructions involved clopen algebras of Ostaszewski spaces, but were couched in purely Boolean algebraic terminology. (Dow independently discovered (c) via a topological approach, but did not circulate his proof.) Retractiveness of Ostaszewski spaces was also of interest to Arhangelskii.

This paper takes a topological view of these questions. Some proofs are shortened, it is easier to tease out the main ideas, it becomes fairly easy to combine or separate them, and their axiomatic strength becomes apparent (because Ostaszewski spaces do not exist

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under, e.g., CH). In particular, a topologically interesting aspect of Shelah's construction turns out to work with no special axioms, and under $\diamondsuit$ there is an Ostaszewski space with all three properties.

The extra effort of these constructions is really needed—under $\diamondsuit$ there is an Ostaszewski space with none of these properties.

We will note in Section 2 that Ostaszewski spaces are Cohen indestructible. The key property that simultaneously gives (a) and (b) is upwards absolute, so Shelah's construction is Cohen indestructible. The $\diamondsuit$ construction for (c) can be jacked up to be Cohen indestructible, i.e., add any number of Cohen reals and the construction retains the desired property.

Theorem A. *Assume $\diamondsuit$. There is an Ostaszewski space which is retractive, homeomorphic to every uncountable closed subspace, and homeomorphic to every locally countable regular Hausdorff uncountable continuous image.*

Remarks. This space can be constructed so that it retains the latter property under the addition of arbitrarily many Cohen reals. In the Boolean algebra setting the extra hypotheses on continuous images are moot, since the Stone space of the clopen algebra (= the 1-point compactification of the Ostaszewski space) is homeomorphic to every Hausdorff continuous image; it turns out that they are necessary in the locally compact setting.

Theorem B. *There is a thin-tall locally compact scattered space X of height ω_1 so that if α is countable then X^α is a retract of X.*

Theorem C. *Assume $\diamondsuit$. There is an Ostaszewski space which is not retractive, not homeomorphic to every uncountable closed subspace, and not homeomorphic to every locally countable regular Hausdorff uncountable continuous image, and which retains these properties when any number of Cohen reals are added.*

These constructions are quite flexible, and can be mixed and matched in more ways than will be done in this paper.

Acknowledging prior sources, the discussion of property (†) below is in large part derived from dualizing Shelah's work, and the Cohen indestructibility trick derives from the proof that under CH there is a Cohen indestructible small almost disjoint family on ω.

1. Preliminaries

Maps and retracts

If $f : X \to Y$ is a continuous map, we define $S_f = \{y \in Y\colon\ |f^{\leftarrow}(y)| \geqslant 2\}$ and $\mathcal{Y}_f = \{f^{\leftarrow}(y)\colon\ y \in S_f\}$.

If Y is a subset of X, we say that Y is a retract iff there is a continuous map $f : X \to Y$ with $f|_Y$ the identity, and we say f is retractive.

X is retractive iff every closed subspace is a retract.

Scattered spaces

For an arbitrary space X, we define the Cantor–Bendixson decomposition as follows: $X^{(0)} = X$; given $X^{(\alpha)}$, $X(\alpha)$ is the set of isolated points of $X^{(\alpha)}$. For each α, $X_\alpha = \bigcup_{\beta<\alpha} X(\beta)$, and $X^{(\alpha)} = X \backslash X_\alpha$. For each x, ht x = the greatest α with $x \in X^{(\alpha)}$. We will frequently write X^α instead of $X^{(\alpha)}$.

A neighborhood u of $x \in X(\alpha)$ is said to be canonical for x iff $u \cap X(\alpha) = \{x\}$.

The CB width of X is the sup of all of the $|X(\alpha)|$'s, and the CB height of X (= ht X) is the first α for which $X(\alpha) = \emptyset$. X is scattered iff $X^{\mathrm{ht}\, X}$ is empty.

The cardinal sequence of a scattered space X is the function $c : \mathrm{ht}\, X \to \mathrm{CARDS}$ where $c(\alpha) = |X(\alpha)|$.

We will use the abbreviation "LCS" to mean "locally compact noncompact scattered Hausdorff space".

A thin-tall space is a scattered space whose CB width is ω and whose CB height is uncountable.

There is a standard way to construct thin-tall LCS spaces of CB height ω_1. Suppose we know the topology on X_α, where α is countable, and we are ready to define the topology on $X_{\alpha+1}$. For each point x in $X(\alpha)$ we assign some U_x, a noncompact countable union of compact sets in X_α so that if $x \neq y$ then $U_x \cap U_y$ is compact. A neighborhood base for x consists of all $\{x\} \cup U_x \backslash K$ where K is compact in X_α.

Important to our constructions will be building the U_x's to meet predetermined infinite sets. Here's how to do this. Suppose E is infinite and closed discrete in X_α, we are building the topology on $X_{\alpha+1}$, we have some fixed $x \in X(\alpha)$ and we want to guarantee that $x \in \mathrm{cl}\, E$ when we're done. Consider the partial order of finite functions from $X(\alpha)$ into {compact open sets in X_α} with pairwise disjoint range, where $\sigma < \tau$ iff $\sigma(y) \supset \tau(y)$ for all $y \in \mathrm{dom}\, \tau$. Then $\{\sigma\colon\ x \in \mathrm{dom}\, \sigma \text{ and } |E \cap \sigma(x)| > n\}$ is dense for all n, x. Applying the Rasiowa–Sikorski lemma completes the construction.

This construction adapts easily to the variants we will use.

Countable scattered spaces

An old result of Mazurkiewicz and Sierpinski says that two countable locally compact spaces are homeomorphic iff they have the same cardinal sequence, i.e., every countable locally compact space is homeomorphic to an ordinal space. Hence every countable compact subset of a locally countable LCS is a retract.

Ostaszewski spaces

An Ostaszewski space is a thin-tall countably compact LCS of height ω_1 in which every closed set is either compact (hence countable) or cocountable. Ostaszewski spaces are hereditarily separable. A space is sub-Ostaszewski iff every closed set is either countable or cocountable. Note that an uncountable continuous image of a countably compact sub-Ostaszewski space is countably compact sub-Ostaszewski.

Cohen forcing

In this paper $\mathbb{Q} = \mathrm{Fn}(\omega, 2)$.

The key technique to getting Cohen indestructibility is a way of diagonizalizing over Cohen conditions to get a ground model set which interacts nicely with a Cohen name for a set. Here is an example. Suppose $\dot{E}$ names a subset of some X_α, we are constructing the topology on $X_{\alpha+1}$, and we want to ensure that if $p \Vdash \dot{E}$ is infinite and closed discrete, then $p \Vdash X(\alpha) \subset \mathrm{cl}\, \dot{E}$. Here is how to do it. Let $\{p_n\colon n < \omega\}$ list $\{p \in \mathbb{Q}\colon p \Vdash \dot{E}$ is infinite closed discrete$\}$ so that every condition is listed infinitely often. Let $\{H_n\colon n < \omega\}$ list every compact open subset of X_α. We construct a set $F = \{y_n\colon n < \omega\}$ as follows: given $\{y_k\colon k < n\}$ we find $q \leqslant p_n$ and $y \notin \{y_k\colon k < n\}$ so $q \Vdash y \in \dot{E} \setminus \bigcup_{k \leqslant n} H_k$. Let $y = y_n$. Then we construct the sets U_x for $x \in X(\alpha)$ so that for every p listed, $\{y_n\colon p = p_n\} \cap U_x$ is finite.

This technique adapts easily to the several variations of it we will invoke.

2. Six propositions and a corollary

When is a closed subset of an Ostaszewski space a continuous image of the space?

Proposition 2.1. *Let X, Y be thin-tall LCS spaces of CB height ω_1, X countably compact, f a continuous function from X to Y with $\bigcup \mathcal{Y}_f$ contained in a compact set. Then X is homeomorphic to Y.*

Proof. Let H be compact with $\bigcup \mathcal{Y}_f \subset H$. Without loss of generality, H is open. Then there is some compact open $K \subset Y$ with $S_f \subset K$. Without loss of generality (CB height $H) \cdot \omega <$ CB height K. Then $f^{\leftarrow}K$ is countable, open, closed, and hence (by countable compactness) compact; since K and $f^{\leftarrow}K$ have the same cardinal invariants, there is a homeomorphism g from $f^{\leftarrow}K$ to K. Let X^* be the one-point compactification of $X \setminus f^{\leftarrow}K$, and Y^* be the one-point compactification of $f[X \setminus f^{\leftarrow}K]$. Since X is countably compact, inverse images of compact subsets of $f[X \setminus f^{\leftarrow}K]$ are compact, so f extends to a homeomorphism between X^* and Y^*. So f is a homeomorphism from $X \setminus f^{\leftarrow}K$ to $f[X \setminus f^{\leftarrow}K]$. Now let $h = f|_{X \setminus f^{\leftarrow}K} \cup g$. Then h is a homeomorphism from X to Y. □

Proposition 2.2. *Suppose X is Ostaszewski and*

(†) *for every countable α, X^α is a retract of X under a function f_α so that $\bigcup\{f_\alpha^{\leftarrow}(x)\colon x$ is isolated in $X^\alpha\} \supset X_\alpha$.*

Then X is retractive and homeomorphic to every closed uncountable subset.

Before proving Proposition 2.2 we prove the following:

Claim. *Suppose X is scattered countably compact first countable, and, for some α, $f: X \to X^\alpha$ is retractive so that $\bigcup\{f^{\leftarrow}(x)\colon x \in X(\alpha)\} \supset X_\alpha$. If u is clopen in X then $\forall^\infty x \in X(\alpha)$ $x \in u$ iff $f^{\leftarrow}(x) \subset u$ iff $f^{\leftarrow}(x) \cap u \neq \emptyset$.*

Proof of Claim. By way of contradiction, suppose there is some infinite $E \subset X(\alpha) \cap u$ so that $\forall x \in E\ f^{\leftarrow}(x) \backslash u \neq \emptyset$. For each $x \in E$ let $y_x \in f^{\leftarrow}(x) \backslash u$. Since X is countably compact first countable there is some w a limit of a convergent sequence of y_x's. Since f is continuous and u is closed, $f(w) \in u$. Since $X \backslash u$ is closed, $w \notin u$. Since $X(\alpha)$ is discrete, $f(w) \in X^{\alpha+1}$, hence $w = f(w)$, contradiction. Similarly, suppose there is some infinite $E \subset X(\alpha) \backslash u$ with $\forall x \in E\ f^{\leftarrow}(x) \cap u \neq \emptyset$. $\forall x \in E$ let $y_x \in f^{\leftarrow}(x) \cap u$. We may assume that $\{y_x\colon x \in E\}$ is a convergent sequence. Then $\lim_{x \in E} y_x = w \in u \cap X^{\alpha}$, so $\lim E = f(w) \in u$, a contradiction. □

Proof of Proposition 2.2. Let Y be an uncountable closed subset of X. By hypothesis $U = X \backslash Y$ is countable, hence contained in some X_α. We may assume that α is large enough so that ht $U \cdot \omega < \alpha$. For each $x \in X(\alpha)$ let $u_x = f_\alpha^{\leftarrow}(x)$. For each $x \in X(\alpha)$, let $Y_x = Y \cap u_x$. Note that $f_\alpha|_{X^\alpha} = \mathrm{id}|_{X^\alpha}$ and each u_x is clopen.

First we show that X is homeomorphic to Y.

Since each u_x has the same cardinal sequence as Y_x, there is a homeomorphism $f_x : u_x \to Y_x$ with $f(x) = x$ for each $x \in X(\alpha)$. Let $f = \mathrm{id}|_{X^\alpha} \cup \bigcup_{x \in X(\alpha)} f_x$. To show that f is continuous it suffices to show that if $v = u \cap Y$, where u is clopen, then $f^{\leftarrow} v$ is open:

$$f^{\leftarrow} v = \left(u \cap X^{\alpha}\right) \cup \bigcup \{u_x\colon\ u_x \subset u\} \cup \bigcup \left\{f_x^{\leftarrow}(u \cap Y_x)\colon\ x \in v \text{ and } u_x \not\subset u\right\}.$$

By the claim and continuity of f_α and of the f_x's this is clopen.

Since f extends to a 1–1 continuous map from the one-point compactification of X to the one-point compactification of Y, f is a homeomorphism.

Now we show that Y is a retract.

Since each u_x is an ordinal space, there is a retractive map $g_x : u_x \to Y_x$. Let $g = \mathrm{id}|_{X^\alpha} \cup \bigcup_{x \in X(\alpha)} g_x$. The proof that g is continuous is similar to the proof that f is continuous. □

Remark. Proposition 2.2 is the topological version of Lemmas 3.6 and 3.8 in [2]; property (†) corresponds to property G3 in that paper: for every α, B has a subalgebra B_α which chooses exactly one element from every equivalence class in $B/\mathcal{I}_\alpha$ so that $\mathcal{I}_\alpha$ is generated by $\bigcup_{b \in B_\alpha(0)} \mathcal{I}_\alpha | b$.

When is a continuous image of an Ostaszewski space homeomorphic to the space?

Proposition 2.3. *Suppose Y is a regular locally countable* 1–1 *continuous image of an Ostaszewski space X. Then Y is locally compact.*

Proof. Let $f : X \to Y$ be continuous 1–1, X Ostaszewski. Y is countably compact sub-Ostaszewski. Since Y is regular, for every point $y \in Y$ and every open neighborhood u of y there is a closed set $K \subset u$ with $y \in \mathrm{int}\, K$. Since Y is locally countable, we may assume K is countable, hence compact. □

Corollary 2.4. *A* 1–1 *continuous function from an Ostaszewski space to an uncountable locally countable regular Hausdorff space is a homeomorphism.*

Proof. Suppose $f: X \to Y$ is continuous 1–1, and X is Ostaszewski. As in the proofs of Propositions 2.1 and 2.2, f extends to a 1–1 continuous map from the one-point compactification of X to the one-point compactification of Y, hence is a homeomorphism. □

Note that "locally countable" is necessary: if X is thin-tall LCS and $x \in X(0)$, let $Y = X \setminus \{x\}$, $Z = Y \cup \{y^*\}$ be the one-point compactification of Y, and let $f: X \to Z$ be given by $f|_Y$ is the identity, $f(x) = y^*$. Then f is continuous, 1–1, but not a homeomorphism.

Proposition 2.5. *Let X be Ostaszewski, $f: X \to Y$, f continuous onto, Y locally countable regular Hausdorff, $\mathrm{cl}\, \mathcal{Y}_f$ compact. Then X is homeomorphic to Y.*

Proof. Let u be compact open with $\mathrm{cl}\, \mathcal{Y}_f \subset u$. If we can find a compact open subset v of Y containing $f[u]$ we're done as in Proposition 2.1: we can assume that v and $f^{\leftarrow v}$ have the same cardinal invariants, and by Corollary 2.4 $X \setminus f^{\leftarrow v}$ is homeomorphic to $Y \setminus v$. Since $f[u]$ is a compact subset of a locally compact 0-dimensional space, such a v exists. □

Thus to ensure that every continuous locally countable regular Hausdorff uncountable image is a homeomorphism, it suffices to show that for every continuous f, $\mathrm{cl} \bigcup \mathcal{Y}_f$ is compact. Here are the tools to do this.

Proposition 2.6. *Suppose $\mathcal{Y}$ is a pairwise disjoint collection of subsets of a countable LCS Z, $\mathrm{cl} \bigcup \mathcal{Y}$ is not compact, and each $Y \in \mathcal{Y}$ has size at least 2. Then there is a set $E = \{\{x_n, y_n\}: n < \omega\}$ so that for all $Y \in \mathcal{Y}$, $x_n \in Y$ iff $y_n \in Y$, and if H is compact in Z then $H \cap \{x_n: n < \omega\}$ is finite.*

Proof. If some $Y \in \mathcal{Y}$ fails to be compact, we are done. So we may assume every $Y \in \mathcal{Y}$ is compact. Z has only countably many compact open sets. Diagonalize through the compact open sets in Z and through $\mathcal{Y}$. □

In the situation of Proposition 2.6, we may assume without loss of generality that either

(I) $\{y_n: n < \omega\}$ converges to a point in Z, or

(II) for every compact H in Z, $H \cap \{y_n: n < \omega\}$ is finite.

In the first case we say that we are in a situation of type I. In the second case we say we are in a situation of type II. In either case we call E an approximation of $\mathcal{Y}$.

Definition. A set of pairs $\{\{x_n, y_n\}: n < \omega\}$ in a scattered space Z is called an approximation iff $\{x_n: n < \omega\}$ is closed discrete, and E satisfies either (I) or (II).

Proposition 2.7. *Suppose X is a hereditarily separable LCS, f is a continuous function on X with uncountable range, and $\mathrm{cl} \bigcup \mathcal{Y}_f$ is not compact. Then*

$$\left\{\alpha:\ \mathrm{cl} \bigcup \{Y \cap X_\alpha:\ Y \in \mathcal{Y}_f\} \text{ is compact}\right\}$$

is countable.

Proof. Otherwise there would be an uncountable increasing sequence of compact sets, which contradicts hereditary separability. □

The next proposition shows that the property of sub-Ostaszewski is somewhat robust. It may be folklore.

Proposition 2.8. *Let X be scattered Hausdorff 0-dimensional, $\mathbb{P}$ a property K notion of forcing. Then X has an uncountable co-uncountable closed subspace iff $V^{\mathbb{P}} \Vdash X$ has an uncountable co-uncountable closed subspace.*

Proof. $\Rightarrow$ is trivial, so we prove $\Leftarrow$. Note that if X has at least two points of Cantor–Bendixson height ω_1 then it has an uncountable co-uncountable closed subspace, so WLOG X is thin-tall and $X^{\omega_1} = \emptyset$.

Suppose $V^{\mathbb{P}} \Vdash X$ has an uncountable co-uncountable closed subspace. Then there is a $\mathbb{P}$-name $\dot{U} = \{\dot{u}_\alpha\colon \alpha < \omega_1\}$ and a $\mathbb{P}$-name $\dot{K} = \{\dot{x}_\alpha\colon \alpha < \omega_1\}$ where $\Vdash_{\mathbb{P}}$ [each $\dot{u}_\alpha$ is countable open and $\dot{u}_\alpha \cap X^\alpha \neq \emptyset$, each $\dot{x}_\alpha \in X_\alpha$, and $\dot{K} \cap \bigcup \dot{U} = \emptyset$].

For each α there is $p_\alpha, v_\alpha, y_\alpha$ with $p_\alpha \Vdash [\dot{u}_\alpha = v_\alpha$ and $\dot{x}_\alpha = y_\alpha]$. By property K there is $E \in [\omega_1]^{\omega_1}$ so if $\alpha, \beta \in E$ then p_α, p_β are compatible. Hence if $\alpha, \beta \in E$ then $y_\alpha \notin v_\beta$. Since each $y_\alpha \notin v_\alpha$, $\bigcup\{v_\alpha\colon \alpha \in E\}$ is open, uncountable, co-uncountable in V. □

Remark. By Proposition 2.8, scattered Hausdorff 0-dimensional sub-Ostaszewski spaces are preserved by Cohen forcing. Since Fleissner has shown that if X is Ostaszewski then adding a Cohen real preserves countable compactness, adding Cohen reals preserves Ostaszewski spaces. On the other hand, Fremlin has constructed a $\diamondsuit$ example of an Ostaszewski space whose countable compactness is destroyed by adding a random real; in this forcing extension the space remains sub-Ostaszewski.

3. Property (†) and Theorem B

In this section we prove Theorem B and set up the main device for combining (†) with other properties.

Suppose we're constructing a thin-tall LCS of CB height ω_1 with property (†). We assume that at stage α there are particularly nice retractive maps, i.e., if $\beta < \alpha$ then there is a retractive map $f_\beta : X_\alpha \to X_\alpha \setminus X_\beta$ with the following property:

(1) *$\bigcup \mathcal{U}_\beta = X_{\beta+1}$, where $\mathcal{U}_\beta = \{f_\beta^{\leftarrow} x\colon x \in X(\beta)\}$.*

Note that any retractive extension $f : X_\gamma \to X_\gamma \setminus X_\beta$ of f_β also has property (1).

At stage α we have X_α and a set disjoint from X_α which we optimistically denote as $X(\alpha)$. We want to construct, for each $x \in X(\alpha)$ an open neighborhood U_x as in Section 1 so that f_β extends to a retractive map $\widehat{f}_\beta : X_{\alpha+1} \to X_{\alpha+1} \setminus X_\beta$. What is required?

Suppose, when we're done, a set $D \subset X_\alpha$ converges to some $x \in X(\alpha)$. To guarantee continuity, we need $\widehat{f}_\beta[D] = f_\beta[D]$ to converge to $\widehat{f}_\beta(x) = x$. The only place we can run into trouble is on $D \cap X_\beta$, so without loss of generality $D \subset X_\beta$. We want:

(2) *If v is a compact open neighborhood of x then $f_\beta[D] \setminus v$ is finite.*

And we can guarantee this if

(3) *There is a compact open neighborhood* w *canonical for* x *so that*

$$(\dagger)_\beta \quad \forall u \in \mathcal{U}_\beta \text{ either } u \subset w \text{ or } u \cap w = \emptyset.$$

Why does (3) imply (2)? For each $y \in D$, let

$$f_\beta(y) = z_y \in X(\beta)$$

and let $u_y = f_\beta^{\leftarrow} z_y$. Let w be as in (3). Since D converges to x, for all but finitely many $y \in D$, $y \in w$. But then, by (3), for all but finitely many z_y, $z_y \in w$. If v is another compact open neighborhood of $x, w\backslash v$ is a compact open set in X_α. If there were some infinite $E \subset D$ so that if $y \in E$ then $y \in v \cap w$ and $z_y \notin v$, then without loss of generality $\{z_y\colon\ y \in E\}$ converges to a point $z \in w\backslash v$. Let $a \subset X_\alpha \cap (w\backslash v)$ be a compact open neighborhood of z. Without loss of generality $f[E] \subset a$. Let $b = f_\beta^{\leftarrow}[a\backslash X_\beta]$. Then b is clopen in X_α, and $b \cap w$ is compact in X_α. But E is a closed discrete infinite subset of $b \cap w$, a contradiction.

In order to guarantee property (3) we need a slightly stronger induction hypothesis:

Induction hypothesis at stage α:

(4) *If* $s \in [\alpha]^{<\omega}$, $\sup s < \gamma < \alpha$, *and* $x \in X(\gamma)$ *then there is a compact open neighborhood* w_s^x *canonical for* x *so that*

$$(\dagger_s) \quad \forall \beta \in s\ \forall u \in \mathcal{U}_{\beta_i} \text{ either } u \subset w_s^x \text{ or } u \cap w_s^x = \emptyset$$

and $X_\gamma \subset \bigcup_{x \in X(\gamma)} w_s^x$.

Note that if w, v satisfy $(\dagger_s)$, so do $w \cup v$, $w \cap v$, and $w\backslash v$.

The task is to construct, for each $s \in [\alpha]^{<\omega}$, and each $x \in X(\alpha)$, a noncompact countable union of compact sets U_s^x satisfying $(\dagger_s)$, and

(5) $\forall x \neq y, s\ \ U_s^x \cap U_s^y = \emptyset$.

(6) $\forall x, s, t\ \ U_s^x \triangle U_t^x$ *is compact in* X_α.

(7) $\forall s\ \bigcup_{x \in X(\alpha)} U_x^x \subset X_\alpha$.

Then the topology where a base for $x \in X(\alpha)$ consists of all $\{x\} \cup U_{\{0\}}^x \backslash K$, where K is compact in X_α, is LCS; each $\{x\} \cup U_s^x$ is compact open; if $x \in X(\alpha)$ then $\mathrm{ht}_{X_{\alpha+1}} x = \alpha$; and, for each $\beta + 1 < \alpha$, the map which extends f_β and is the identity on the $X(\alpha)$'s is retractive.

Note that for fixed σ any map sending U_s^x to x for all $x \in X(\alpha)$ is retractive, so we can define f_α at the next stage by $f_\alpha[U_{\{0\}}^x] = \{x\}$, and the induction hypothesis still holds.

It remains, then, to construct the U_s^x's. This is done using finite approximations and the Rasiowa–Sikorski lemma (see, e.g., [4]), as follows:

Let $\Sigma = [\alpha]^{<\omega}$. Consider the partial order consisting of all finite functions u from $X(\alpha) \times \Sigma$ where each $u(x, s)$ (denoted u_s^x) is compact in X_α, satisfies $(\dagger_\beta)$ for each $\beta \in s$, and if $x \neq y$ then $u_s^x \cap u_s^y = \emptyset$. The order is: $v \leqslant u$ iff for each x, s, v_s^x extends u_s^x and

(8) *if* $(x, s), (x, t) \in \operatorname{dom} u$ *then* $v_s^x \backslash (u_t^x \cup u_s^x) = v_t^x \backslash (u_t^x \cup u_s^x)$.

We define $S_u = \{s\colon\ \exists x (x, s) \in \operatorname{dom} u\}$, $T_u = \{x\colon\ \exists s (x, s) \in \operatorname{dom} u\}$.

Let's call this partial order $\mathbb{D}$. The idea is to find countably many dense sets in $\mathbb{D}$ so that if G is a filter meeting them each $U_s^x = \bigcup \{u_s^x\colon\ u_s^x \in G\}$ is noncompact (hence nonempty) and has properties (5), (6) and (7). Properties (5) and (6) are immediate from the definition of the partial order. To show that each U_s^x is noncompact, for each $x \in X(\alpha)$, $s \in \Sigma$, and H compact open in X_α, consider $\{u \in \mathbb{D}\colon u_s^x \backslash H \neq \emptyset\}$. By property (4), this set is dense.

To show that property (7) holds, fix $y \in X(\beta)$ for some $\beta < \alpha$, $s \in \Sigma$, and consider $D = \{u \in \mathbb{D}\colon \exists x\ y \in u_s^x\}$. We need to show that this set is dense.

Fix u. Without loss of generality we may assume $s \in S_u$ and $u \notin D$. There are three cases to consider.

Case 1: $\beta \in s$. Let $\gamma > \sup s$, $\gamma \notin \bigcup S_u$, $z \in X(\gamma)$ *with* $y \in w_s^z$. If $\exists x \in T_u$ with $w_s^z \cap u_s^x \neq \emptyset$, extend u to v where

$$v_s^x = (u_s^x \cup w_s^z) \backslash \bigcup_{\bar{x} \in T_u \backslash \{x\}} u_s^{\bar{x}}.$$

Otherwise let $x \notin T_u$ and extend u to v where $v_s^x = w_s^z$.

Case 2: $\beta \notin s$, $\beta < \sup s$. Let $\bar{s} = s \cup \{\beta\}$, $\sup s < \gamma < \alpha$, $z \in X(\gamma)$ such that $y \in w_s^z$. Proceed as in Case 1, substituting $\bar{s}$ for s, and letting $v_{\bar{s}}^x = v_s^x$.

Case 3: $\beta > \sup s$. If $\exists x \in T_u\ w_s^y \cap u_s^x \neq \emptyset$, extend u to v where

$$v_s^x = u_s^x \cup w_s^y \backslash \bigcup_{\bar{x} \in T_u \backslash \{x\}} u_s^{\bar{x}}.$$

Otherwise let $x \notin T_u$ and extend u to v where $v_s^x = w_s^y$.

The proof of Theorem B is, then, as follows: Note that when $\alpha = 1$ the induction hypothesis holds automatically. At each countable stage follow the above construction.

Note that the proof of Theorem B makes no constraints on how we construct new convergent sequences from old closed discrete sets. Given an arbitrary infinite closed discrete $E \subset X_\alpha$ we could consider for each x, s, n, $\overline{D} = \{u\colon |u_s^x \cap E| > n\}$. A proof similar to the proof that property (7) holds shows that $\overline{D}$ is dense in $\mathbb{D}$.

An easy exercise is to show that under CH there is a countably compact thin-tall LCS with property (†). A further exercise is to adapt the method of Section 5 to construct such a space under CH which is Cohen indestructible. (Note that (†) is upwards absolute, so of course Cohen indestructible—it's making countably compact Cohen indestructible that takes a little work.)

4. (†) and Ostaszewski

Suppose we want to get an Ostaszewski space X under $\diamondsuit$. We will construct X so that its underlying set is $\omega \times \omega_1$, and each $X(\alpha) = \omega \times \{\alpha\}$. Let $\{S_\alpha\colon \alpha < \omega_1\}$ be a $\diamondsuit$-sequence for subsets of $\omega \times \omega_1$, i.e.,

(9) *each* S_α *is a subset of* $\omega \times \alpha$, *and*

(10) *if* E *is a subset of* $\omega \times \omega_1$, *then* $\{\alpha\colon E \cap \omega \times \alpha = S_\alpha\}$ *is stationary.*

If S_α has compact closure in X_α just extend to $X_{\alpha+1}$ arbitrarily. Otherwise, let S be an infinite closed discrete subset of S_α and ensure that $X(\alpha) \subset \mathrm{cl}_{X_\alpha} S$. Then $\mathrm{cl}_X S$ will be cocountable. $\diamondsuit$ reflection completes the proof.

Now we take (†) into account. By the remark after the proof of Theorem B, when faced with constructing the topology so $X(\alpha) \subset \operatorname{cl} S$, $S \subset S_\alpha$, S closed discrete in X_α, and we want (†) to hold, ensure $S \cap U_s^x$ is infinite for each $x \in X(\alpha)$, $s \in \Sigma$.

5. Continuous images

Assume $\diamondsuit$. Let's suppress (†) for a moment, and suppose we want $\mathcal{Y}_f$ to have compact closure for every continuous f. At stage α, let $\mathcal{Y}$ be as in the hypothesis of Proposition 2.3. Consider the αth approximation $E = \{\{x_n, y_n\}: n < \omega\}$ as in the conclusion of Proposition 2.3. If E is of type I, we ensure that every point in $X(\alpha)$ is in the closure of $\{x_n: n < \omega\}$. If E is of type II, we ensure that $\{x_n: n < \omega\}$ converges to a single point of $X(\alpha)$ and that every other point of X_α is in the closure of $\{y_n: n < \omega\}$. In either case, if $\mathcal{Y} = \{Y \cap X_\alpha: Y \in \mathcal{Y}_f\}$, then f must have countable range.

At this point we need to take Cohen indestructibility into account, since it is no longer automatic. (We are still not ready for (†).)

At stage α we have X_α and a $\mathbb{Q}$-name for a countable collection $\dot{\mathcal{Y}}$ of disjoint subsets of X_α so that $1_{\mathbb{Q}} \Vdash \bigcup \dot{\mathcal{Y}}$ is not compact. By Proposition 2.3 there is a name $\dot{E} = \dot{E}_\alpha$ for an approximation of $\dot{\mathcal{Y}}$.

By diagonalizing we construct sequences $\{z_n: n < \omega\}$, $\{w_n: n < \omega\}$ so that

(12) *If H is compact in X_α then $\forall^\infty n\ z_n \notin H$, and if $p \Vdash \dot{E}$ is of type II, then $p \Vdash \{n: w_n \notin H\}$ is infinite.*

(13) *If $p \in \mathbb{Q}$ and either $p \Vdash \dot{E}$ is of type I, or $p \Vdash \dot{E}$ is of type II then $p \Vdash \{n: \{z_n, w_n\} \in \dot{E}\}$ is infinite.*

Fix some $x^* \in X(\alpha)$. The goal is to construct, for each $x \in X(\alpha)$ a set U_x as in Section 1 so that for all x, p,

(14) *if $p \Vdash \dot{E}$ is of type I, then $p \Vdash \{k: z_k \in U_x$ and $\{z_k, w_k\} \in \dot{E}\}$ is infinite.*

(15) *if $p \Vdash \dot{E}$ is of type II, then $p \Vdash \{k: z_k \in U_{x^*}, w_k \in U_x$, and $\{z_k, w_k\} \in \dot{E}\}$ is infinite.*

Conditions (14) and (15) ensure that if $\dot{E}$ names an approximation of some $\mathcal{Y}_f$ then $\operatorname{dom} \dot{f}$ is countable.

Note that conditions (14) and (15) can be met, by the key technique, for any $\dot{E}$ naming an approximation and still be meshed with the Ostaszewski construction. CH suffices to consider all such $\dot{E}$. Note also that by satisfying conditions (14) and (15) for all $\dot{E}$ naming an approximation, there are no retractive maps $\dot{f}$ with $\bigcup \mathcal{Y}_{\dot{f}}$ not compact, in particular no X^α is a retract. By relaxing the Ostaszewski requirement to one of countable compactness this gives

Theorem D. *Assume CH. There is a countably compact thin-tall LCS space so that each of its retractive maps f has $\operatorname{cl} \bigcup \mathcal{Y}_f$ compact, and these properties are preserved when adding arbitrarily many Cohen reals.*

Note that the space of Theorem D is, by Proposition 2.1, homeomorphic to each of its thin-tall LCS continuous images.

By restoring the Ostaszewski requirement we have

Theorem E. *Assume $\diamondsuit$. There is an Ostaszewski space homeomorphic to every locally countable regular Hausdorff uncountable continuous image so each of its retractive maps f has compact $\operatorname{cl}\bigcup\mathcal{Y}_f$, and which retains these properties when any number of Cohen reals are added.*

Proof. By Proposition 2.4, if X is Ostaszewski and $\dot{f}: X \to \dot{Y}$ continuous and $\operatorname{cl}\bigcup\mathcal{Y}_{\dot{f}}$ is not compact then $\dot{f}$ has an approximation $\dot{E}$. Putting this together with Proposition 2.5 and the previous construction completes the proof. (Note that $\diamondsuit$ was only used to get Ostaszewski, i.e., to allow us to invoke Propositions 2.4 and 2.5.)

Of course the construction of Theorem E destroys too much. It has no nontrivial retracts; in particular, no X^α is a retract for $\alpha > 0$. The proof of Theorem A requires us to judiciously choose our $\dot{E}$'s so that we don't preclude (†). How can we do this?

Definition 5.1. Let $g: X \to Z$ be continuous onto, where X in thin-tall, and let $f: X \to X^\alpha$ be retractive so that $X_\alpha \subset \bigcup\{f^{\leftarrow}x\colon\ x \in X(\alpha)\}$. We say that g is derived from f iff $g|_{X^\alpha}$ is a homeomorphism onto Z^α, $f[X_\alpha] = Z_\alpha$, and the map $h: Z \to Z^\alpha$ is a retract, where h is defined by: $h|_{Z^\alpha}$ is the identity and, for each $z \in Z(\alpha)$, there is $x \in X(\alpha)$ with $h^{\leftarrow}z = g[f^{\leftarrow}x]$.

Proposition 5.2. *Let $g: X \to Z$ be continuous, g derived from some retractive $f: X \to X^\alpha$ so that $X_\alpha \subset \bigcup\{f^{\leftarrow}x\colon\ x \in X(\alpha)\}$. Then X and Z are homeomorphic.*

Proof. Given g, f as in the hypothesis, let h be as in the definition. Note that if $h^{\leftarrow}z = g[f^{\leftarrow}x]$ then the cardinal invariants of $f^{\leftarrow}x$ and $h^{\leftarrow}z$ are the same, so there is a homeomorphism $k_x: f^{\leftarrow}x \to h^{\leftarrow}z$. As in Proposition 2.2, we define $k: X \to Z$ by $k|_{X^\alpha} = g|_{X^\alpha}$ and, for each $x \in X(\alpha)$, $k|_{f^{\leftarrow}x} = k_x$. The claim used for Proposition 2.2 holds for both X, f and Z, h, so a proof similar to the proof of Proposition 2.2 works. □

To prove Theorem A, then (and suppressing the details about Cohen indestructibility), we only consider $\dot{E}$ judiciously chosen from a $\diamondsuit$ sequence $\{\dot{E}_\alpha\colon\ \alpha < \omega_1\}$ for approximations of ω-sequences into $[\omega \times \omega_1]^2$ as follows. At stage α we work with $\dot{E} = \dot{E}_\alpha$. We ask if some co-finite subset of $\dot{E}_\alpha$ approximates a function derived from any f_β, $\beta < \alpha$, where the f_β's are the witnesses for (†) already constructed. We only go ahead and consider $\dot{E}_\alpha$ if the answer is no. In this case we satisfy (14) and (15). By Proposition 2.4 and a Lowenheim–Skolem and $\diamondsuit$ argument, if $g: X \to Z$ is continuous, Z is uncountable, and X, Z are not homeomorphic, then there is some α so that $\dot{E}_\alpha$ is an approximation of g and no cofinite subset of $\dot{E}_\alpha$ approximates a function derived from any f_β, $\beta < \alpha$, which completes the proof.

6. None of the above

The task here is to construct an Ostaszewski space X in which $X^{(1)}$ is a retract which fails to be homeomorphic to X, and $X^{(2)}$ is neither a retract nor homeomorphic. Since we don't have to worry about (†), the construction is more straightforward. The interested reader can modify the construction to get an Ostaszewski space in which property (†) holds but which is not homeomorphic to one of its uncountable continuous images.

To get $X^{(1)}$ a retract, let $\{u_x\colon x \in X(1)\}$ be an arbitrary clopen partition of X_2 so that each $x \in u_x$. Require that every $z \in X^{(2)}$ has a compact neighborhood w so that, for all $x \in X(1)$, $u_x \cap w \neq \emptyset$ iff $u_x \subset w$. This is compatible with the Ostaszewski construction.

Suppose $\dot{f}$ is a $\mathbb{Q}$-name for a partial function from some $X(\beta)$ to some $X(\delta)$, where $\beta < \delta$, and suppose we are at stage α in our construction of X, $\delta < \alpha$. To ensure that $\dot{f}$ does not extend to a continuous map on X we create a sequence E in $X(\beta)$ so that $\dot{f}[E]$ converges to a point in $X_{\alpha+1}$ but $X(\alpha) \subset \mathrm{cl}_{X_{\alpha+1}} E$.

Again, there is no conflict with this sort of requirement and the requirements used in building Ostaszewski spaces.

In particular, we let $\{\dot{f}_\alpha\colon \alpha < \omega_1\}$ be a CH sequence for approximations of all names of countable functions from $\omega \times \omega_1$ into itself. At stage α we ask if $\dot{f}_\alpha$ names a 1–1 onto function from $X(0)$ to $X(1)$ and, if it does, we ensure that it does not extend to a continuous function at the next level. Note that this does not conflict with the requirement that makes $X^{(1)}$ a retract, and it takes care of all potential homeomorphisms from X to $X^{(1)}$.

If $\dot{f}_\alpha$ does not name such a function, we ask if it names a partial function from $X(0)$ to $X(2)$ and, if it does, we ensure that it does not extend to a continuous function at the next level. Again, there is no conflict with any previous requirement. Since we are looking at partial functions, and since if $f : X \to X^{(2)}$ is a retractive map then, for each $x \in X(2)$, $f^{\leftarrow}x$ is clopen, hence intersects $X(0)$, this takes care of all potential retractive maps as well as all potential homeomorphisms from X to $X^{(2)}$.

Note that without requiring Ostaszewski we only needed CH, i.e.,

Theorem F. *Under CH there are Cohen indestructible thin-tall countably compact spaces which are not retractive, not homeomorphic to every closed subspace, and not homeomorphic to every locally countable regular Hausdorff continuous image.*

7. Questions

As we gave seen, these constructions can be mixed and matched in various ways, but some open questions remain:

Question 1. If an Ostaszewski space is retractive, is it homeomorphic to each of its uncountable closed subspaces?

Question 2. If an Ostaszewski space is homeomorphic to each of its uncountable closed subspaces, is it retractive?

Question 3. Does CH imply the existence of a thin-tall LCS which is homeomorphic to each of its uncountable locally countable regular Hausdorff continuous images?

Remark. Under CH there is a Kunen line which is homeomorphic to each of its uncountable closed subspaces [7].

A variation of Question 3 is

Question 4. Is there a consistent example of a non-Ostaszewski space which is homeomorphic to each of its uncountable locally countable regular Hausdorff continuous images?

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